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A partial unification model in non-commutative geometry

B.E. Hanlon, G.C. Joshi

Research Centre for High Energy Physics, School of Physics, University of Melbourne, Parkville, Victoria 3052, Australia

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Abstract

We consider the construction of $SU(2)_L \otimes SU(2)_R \otimes SU(4)$ partial unification models as an example of phenomenologically acceptable unification models in the absence of supersymmetry in non-commutative geometry. We exploit the Chamseddine, Felder and Fröhlich generalization of the Connes and Lott model building prescription. By introducing a bi-module structure and appropriate permutation symmetries we construct a model with triplet Higgs fields in the $SU(2)$ sectors and spontaneous breaking of $SU(4)$.

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1. Introduction

As with other extensions of space–time, non-commutative geometry provides a framework in which scalar Higgs fields may be introduced on the same level as gauge fields. In higher dimensional models, Higgs fields result from gauge fields which originally carried space indices corresponding to the, now compactified, additional dimensions. While this procedure has an aesthetic appeal, phenomenological problems arise from the existence of a single order parameter associated with the compactification scale, usually taken at the Planck scale [1]. Non-commutative geometry provides an alternative framework in which differing scales may exist.

These geometrical considerations emerge from applications of gauge theory beyond Riemannian spaces. The notion of a manifold is generalized to be the

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product of a continuous manifold by a discrete set of points. Gauge fields now arise from appropriately chosen fibre bundles along the continuous directions, while Higgs scalars result from gauging the discrete directions. Since spinor fields are the fundamental fields in non-commutative gauge theory, the fermionic action can be introduced in a simple way. Consequently, realistic phenomenological models can be considered within this model building prescription and indeed the standard model has been made the subject of this approach [2].

If these notions are to be applied to GUT models then the original model building prescription of Connes and Lott must be reformulated [2–4]. This allows for gauge theories which are not constrained to be product symmetries. The reformulation, introduced by Chamseddine, Felder and Fröhlich [5], consists of embedding the symmetry breaking in the Dirac operator such that gauge invariance is not broken. This simplifies and generalizes model building and allows for the introduction of permutation symmetries between copies of space–time, yielding Higgs representations necessary for symmetry breaking at appropriate scales.

The $SU(5)$ GUT model constructed by this approach did not provide any additional suppression on the rate of proton decay and therefore is ruled out experimentally. Appeals to space–time supersymmetry in non-commutative geometry have yet to be formally developed and do not appear to have an obvious answer. For this reason other avenues must be explored in order to yield acceptable models. Such examples are provided by GUT models with extended symmetry breaking schemes, such as $SO(10)$. As well as suppression of the proton decay rate, such quark–lepton unified models allow for the consistent inclusion of a right handed neutrino and the freedom to incorporate other phenomenological features, such as a reasonable value for $\sin^2\theta_W$ [6]. Originally it was speculated that the Higgs fields required to implement such a scheme within non-commutative geometry could not be easily constructed within the model building prescription and would need to be added as an external field, not associated to any vector [5]. Recently, however, Chamseddine and Fröhlich succeeded in constructing a consistent $SO(10)$ model [7]. This represents an important step in the development of a deeper understanding behind the possible origin of mass scales in such extended models by generalizing the permutation symmetry between space–times to include conjugation symmetries as well as direct identifications. In order to realize an acceptable model, however, it was necessary to introduce additional singlet spinors so that Higgs fields transforming as **16**'s could be included, yielding Cabibbo angle mixing among down quarks.

While the $SO(10)$ breaking scheme is not unique, the spontaneous breaking pattern realizing the Pati–Salam partial unification $SU(2)_L \otimes SU(2)_R \otimes SU(4)$ has an appealing symmetry in which the phenomenological featues are most transparent [8]. Importantly, this is also the minimal symmetry group incorporating both quark–lepton unification and the quantization of electric charge. The electroweak sector of such left–right symmetric theories has already been investigated by Chamseddine et al. yielding spontaneous symmetry breaking of $SU(2)_R$ by triplets [5]. Other approaches to non-commutative geometry, originally considered by Coquereaux et al. [9], Dubois-Violette et al. [10] and Balakrishna et al. [11], have also been generalized to yield a left–right symmetric weak interaction model, this time with doublet Higgs fields [12]. It was found that this made maximal use of the gauge connection in the discrete directions. While these alternative model building prescriptions will not be pursued here, they help to demonstrate a natural identification of left–right symmetric models with non-commutative geometry.

In this paper we wish to extend this investigation of left–right symmetric models to the Pati–Salam unification symmetry. Exploiting the model building approach of Chamseddine et al. [5] we will first consider a minimal model and then explicitly construct a model with triplet Higgs in the $SU(2)$ sectors and spontaneous breaking of $SU(4)$. To realize such a scheme we will include a bi-module structure similar to that for the addition of $SU(3)$ colour to the standard model [2]. However, unlike in the $SU(3)$ case, we do not wish the symmetry breaking matrices in these directions to be identically zero, thus introducing a non-trivial extension. We find that a very natural model emerges without the need to introduce singlet spinors or an additional set of conjugate fermions to produce coupling to conjugate bi-doublet Higgs.

2. The model building prescription

We wish to give an overview of the model building prescription outlined by Chamseddine et al. [5]. The geometrical setting is that of Connes [4], with the reformulation that the choice of gauge structure is defined directly within the Dirac operator. This is in contrast to the original prescription of Connes and Lott [2] where the gauge structure results from the choice of vector bundle $\mathcal{E}$, defined as a finite projective right module over the algebra $\mathcal{A}$ defining the non-commutative space. It follows that the natural choice of vector bundle must now be $\mathcal{E} = \mathcal{A}$ i.e. the orthogonal projection is trivial. With this choice the connection and curvature have the simplest form.

The notion of geodesic distance is incorporated in the concept of a K-cycle. A K-cycle over the involutive algebra $\mathcal{A}$ is a $*$-action of $\mathcal{A}$ by bounded operators on a Hilbert space $\mathcal{H}$ and a possibly unbounded, self-adjoint operator D, denoted Dirac operator, such that $[D, f]$ is a bounded operator for all $f \in \mathcal{A}$ and $(1 + D^2)^{-1}$ is compact. As we will be interested in four point spaces we will cast definitions about this choice although they are easily extended to any number of points. Let X be a compact Riemannian spin manifold, $\mathcal{A}_1$ the algebra of functions on X and $(\mathcal{H}_1, D_1)$ the Dirac K-cycle with $\mathcal{H}_1 = L^2(X, \sqrt{g}\, d^d x)$ on $\mathcal{A}_1$. Denote by γ_5 the fifth anticommuting Dirac gamma matrix, the chirality operator, given by $\gamma_5 = \gamma_1\gamma_2\gamma_3\gamma_4$ in four dimensional Euclidean space, defining a Z_2 grading on $\mathcal{H}_1$. Let $\mathcal{A}_2$ be given by $\mathcal{A}_2 = M_n(C) \oplus M_p(C) \oplus M_q(C) \oplus M_r(C)$, where $M_n(C)$ is the set of all $n \times n$ matrices, with the K-cycle $(\mathcal{H}_2, D_2)$ and $\mathcal{H}_2 = \mathcal{H}_n \oplus \mathcal{H}_p \oplus \mathcal{H}_q \oplus \mathcal{H}_r$ corresponding to the Hilbert spaces C^n, C^p, C^q and C^r respectively. The product geometry is then given by

$$\mathcal{A} = \mathcal{A}_1 \otimes \mathcal{A}_2 \ , \tag{1}$$

with the Dirac operator correspondingly written as

$$D = D_1 \otimes 1 \otimes 1 + \gamma_5 \otimes D_2 \ , \tag{2}$$

D_2 being comprised of tensor products acting on the four point Hilbert space $\mathcal{H}_2$. The decomposition of $\mathcal{H}_2$ diagonalizes the action of $f \in \mathcal{A}$

$$f \to \mathrm{diag}(f_1, f_2, f_3, f_4) \ .$$

The operator D is then

$$D = \begin{pmatrix} \not{\partial} \otimes 1 \otimes 1 & \gamma_5 \otimes M_{12} \otimes K_{12} & \gamma_5 \otimes M_{13} \otimes K_{13} & \gamma_5 \otimes M_{14} \otimes K_{14} \\ \gamma_5 \otimes M_{21} \otimes K_{21} & \not{\partial} \otimes 1 \otimes 1 & \gamma_5 \otimes M_{23} \otimes K_{23} & \gamma_5 \otimes M_{24} \otimes K_{24} \\ \gamma_5 \otimes M_{31} \otimes K_{31} & \gamma_5 \otimes M_{32} \otimes K_{32} & \not{\partial} \otimes 1 \otimes 1 & \gamma_5 \otimes M_{34} \otimes K_{34} \\ \gamma_5 \otimes M_{41} \otimes K_{41} & \gamma_5 \otimes M_{42} \otimes K_{42} & \gamma_5 \otimes M_{43} \otimes K_{43} & \not{\partial} \otimes 1 \otimes 1 \end{pmatrix} \tag{3}$$

where M_{mn}, $m \neq n$, is an $m \times n$ complex matrix such that $M_{mn}^\dagger = M_{nm}$ and each K_{mn} is a 3×3 family mixing matrix. The M_{mn} correspond to the tree level vacuum expectation values of Higgs fields, the chosen form of which determines the symmetry breaking pattern.

The space of forms $\Omega^*(\mathcal{A}) = \oplus_{n=0}^{\infty} \Omega^n(\mathcal{A})$ is generated by elements $a_0\, da_1 \cdots da_k \in \Omega^k(\mathcal{A})$ such that $a_0, a_1, \ldots \in \mathcal{A}$. With $\mathcal{E} = \mathcal{A}$ a connection is given by the element

$$\rho = \sum_i a^i db^i \in \Omega^1(\mathcal{A}) \ , \tag{4}$$

with the curvature specified by

$$\Theta = d\rho + \rho^2 \in \Omega^2(\mathcal{A}) \tag{5}$$

where $d1 = 0$ and the ρ^2 term does not vanish. An involutive representation of $\Omega^*(\mathcal{A})$ by bounded operators on $\mathcal{H}$, with algebra $B(\mathcal{H})$, is defined by the map $\pi : \Omega^*(\mathcal{A}) \to B(\mathcal{H})$ given by

$$\pi(a_0\, da_1 \cdots da_n) = a_0 [D, a_1][D, a_2] \cdots [D, a_n] \ . \tag{6}$$

Consequently

$$\pi(\rho) = \sum_i a^i [D, b^i] \ . \tag{7}$$

Evaluating (7) yields the result

$$\pi(\rho) = \begin{pmatrix} A_1 & \gamma_5 \otimes \phi_{12} \otimes K_{12} & \gamma_5 \otimes \phi_{13} \otimes K_{13} & \gamma_5 \otimes \phi_{14} \otimes K_{14} \\ \gamma_5 \otimes \phi_{21} \otimes K_{21} & A_2 & \gamma_5 \otimes \phi_{23} \otimes K_{23} & \gamma_5 \otimes \phi_{24} \otimes K_{24} \\ \gamma_5 \otimes \phi_{31} \otimes K_{31} & \gamma_5 \otimes \phi_{32} \otimes K_{32} & A_3 & \gamma_5 \otimes \phi_{34} \otimes K_{34} \\ \gamma_5 \otimes \phi_{41} \otimes K_{41} & \gamma_5 \otimes \phi_{42} \otimes K_{42} & \gamma_5 \otimes \phi_{43} \otimes K_{43} & A_4 \end{pmatrix} \tag{8}$$

the A's and ϕ's are determined in terms of the a's and b's by

$$\begin{aligned} A_m &= \sum_i a^i_m \not{\partial} b^i_m \ , \\ \phi_{mn} &= \sum_i a^i_m (M_{mn} b^i_n - b^i_m M_{mn}) \end{aligned} \tag{9}$$

satisfying $A^\dagger_m = A_m$ and $\phi^\dagger_{mn} = \phi_{nm}$. The two form $d\rho = \sum_i da^i\, db^i$ with image under π given by $\pi(d\rho) = \sum_i [D, a^i][D, b^i]$ can be similarly evaluated.

Unitary gauge transformations by $g \in U(\mathcal{A}) = \{g \in \mathcal{A} : g^\dagger g = 1\}$ can be defined in terms of transformations on the a^i and b^i such that

$$\begin{aligned} a^i &\to {}^g a^i = g a^i \ , \\ b^i &\to {}^g b^i = b^i g^\dagger \ . \end{aligned} \tag{10}$$

This definition implies the constraint

$$\sum_i a^i b^i = 1 \ , \tag{11}$$

which can be imposed without loss of generality. It is straightforward to compute the action of gauge transformations on $\pi(\rho)$ which in component form can be written as

$$\begin{aligned} {}^g A_m &= g_m A_m g^\dagger_m + g_m \not{\partial} g^\dagger_m \ , \\ {}^g(\phi_{mn} + M_{mn}) &= g_m (\phi_{mn} + M_{mn}) g^\dagger_n \ . \end{aligned} \tag{12}$$

Thus the A_m are the gauge fields while $\phi_{mn} + M_{mn}$ are scalar fields transforming covariantly. The ϕ_{mn} represent fluctuations around the vacuum state so that we are in fact working in the spontaneously broken phase for which the Higgs potential will be minimized when $\phi_{mn} = 0$.

A crucial aspect which must be considered is that the representation π is ambiguous, with the correct space of forms actually given by $\Omega^*(\mathcal{A})/ \operatorname{Ker}\pi + d\operatorname{Ker}\pi$ [3]. Working on $\Omega^*(\mathcal{A})$ will result in the appearance of auxiliary fields into which the scalar Higgs potential could be absorbed, removing the Higgs mechanism from the model. The potential is saved from disappearing, however, by including the 3×3 family mixing matrices K_{mn}. Nevertheless, in calculating the potential it is necessary to determine which of the auxiliary fields are truly independent. If all the auxiliary fields are independent the Higgs potential will disappear regardless. This places severe constraints on model building and the choice of vacuum expectation values for which the independence or otherwise of the auxiliary fields will depend.

Since the fermionic fields are the fundamental fields, the spinor action can be expressed simply as

$$I_\Psi = \langle \Psi, (D + \pi(\rho))\Psi \rangle \ . \tag{13}$$

To determine the Yang–Mills action the notion of Dixmier trace must be considered. The action is given by the positive definite expression

$$I = 1/8\, \mathrm{Tr}_w (\Theta^2 |D|^{-4}) \ , \tag{14}$$

where the Dixmier trace is defined by

$$\mathrm{Tr}_w(|T|) = \lim\nolimits_w \frac{1}{\log N} \sum_i^N \mu_i(T) \tag{15}$$

for a compact operator T and eigenvalues μ_i of $|T|$. For the Dirac operator the action can be equivalently expressed as

$$I = \frac{1}{8} \int d^4x \ \mathrm{Tr}(\mathrm{tr}(\pi^2(\Theta))) \ , \tag{16}$$

where tr is over the Clifford algebra and Tr is over the matrix structure. Finally, the action is analytically continued to Minkowski space.

3. A minimal model

By minimal, we are making reference to a model for which the simplest Higgs sector can be constructed to implement the required symmetry breaking scheme. This is in analogy to the minimal $O(10)$ model of Witten [13]. We will consider a Riemannian spin manifold extended by four points with the algebra given by

$$\mathcal{A}_2 = M_2(C) \oplus M_4(C) \oplus M_4(C) \oplus M_2(C) \ , \tag{17}$$

together with the permutation symmetry

$$a_2^i = a_3^i, \qquad b_2^i = b_3^i \ .$$

In this way the second and third copies are identified and we have Higgs fields transforming in a self adjoint rather than a product representation in this region. With this choice the vector potential $\pi(\rho)$ becomes

$$\pi(\rho) = \begin{pmatrix} A_L & \chi_L & \chi_L & \phi \\ \chi_L^\dagger & A_4 & \Sigma & \chi_R^\dagger \\ \chi_L^\dagger & \Sigma & A_4 & \chi_R^\dagger \\ \phi^\dagger & \chi_R & \chi_R & A_R \end{pmatrix} \tag{18}$$

where the gauge fields $A = \gamma^\mu A_\mu$ are self adjoint $n \times n$ gauge vectors, Σ is a self adjoint 4×4 scalar field (i.e. $\Sigma_{23} = \Sigma_{32} = \Sigma_{32}^\dagger$), χ_L and χ_R are 2×4 complex scalar fields and ϕ is a bidoublet scalar field. A_L, A_R and A_4 are $U(2)_L, U(2)_R$ and $U(4)$ gauge fields respectively.

Note that the Pati–Salam partial unification has no $U(1)$ symmetries. Thus in the reduction from $U(2)_L \otimes U(2)_R \otimes U(4)$ to $SU(2)_L \otimes SU(2)_R \otimes SU(4)$ we do not need to relate or introduce $U(1)$ factors. To induce the reduction we impose the constraint

$$\mathrm{Tr}(A_L + A_R) = 2\,\mathrm{Tr}(A_4) = 0 \ , \tag{19}$$

reducing $U(2)_L \otimes U(2)_R$ to $SU(2)_L \otimes SU(2)_R$ and $U(4)$ to $SU(4)$. It is important to stress that these permutation symmetries and trace conditions are constraints introduced by hand in order to yield phenomenologically credible results. Σ will now introduce spontaneous breaking of $SU(4)$ to $SU(3) \otimes U(1)_{B-L}$, $\chi_{L,R}$ allows for asymmetric breaking of $SU(2)_{L,R}$ by appropriate choices of vacuum expectation values while ϕ is responsible for symmetry breaking at the electroweak scale.

Introducing the fermionic sector now poses a dilemma. The multiplet structure for one family is

$$\psi_{L,R} = \begin{bmatrix} u_r & u_b & u_g & u_l = \nu_e \\ d_r & d_b & d_g & d_l = e^- \end{bmatrix}_{L,R} \tag{20}$$

where lepton number is identified as the fourth colour. The representation structure with respect to $SU(2)_L \otimes SU(2)_R \otimes SU(4)$ is $\psi_L = (\mathbf{2}, \mathbf{1}, \mathbf{4})$ and $\psi_R = (\mathbf{1}, \mathbf{2}, \mathbf{4})$. Clearly, there are no gauge invariant couplings of these fields with the Higgs scalars χ_L and χ_R. Coupling of fermions to the bidoublet field ϕ is responsible for the generation of the usual quark and lepton masses but the generation of a heavy right handed neutrino depends on an extended interaction sector. This neutrino can get such a mass only through mixing with exotic fermions [14]. This is difficult to implement in this scheme beyond the inclusion of fermionic singlets. Note also that we cannot appeal to higher order effects as we are dealing with classical geometries. A viable model without exotics must, therefore, induce the required breaking and mass generation at tree level from Higgs scalars corresponding to the standard fermions only. That is we require a non-minimal model.

It is worthwhile pointing out that with no identification between the $SU(2)$ and $SU(4)$ sectors that fermions corresponding to these different copies of space–time will not, in fact, have the required representation structure. That is, we would have fermions transforming in the fundamental representation **2** of $SU(2)$ which are singlets under $SU(4)$ and independent fermions in the **4** of $SU(4)$ which are $SU(2)$ singlets, rather than with the multiplet structure (20). While it may be possible to write down an appropriate spinor Ψ to implement the required representations the necessary identifications would have to be imposed in an ad-hoc way, external to the model building program. This problem of identification is related to the indifference of the $SU(4)$ fermion representations on their chirality.

4. A non-minimal model

An examination of the vector potential (18) demonstrates that it is not possible to have Higgs scalars transforming as a product representation with one component the adjoint of a chosen symmetry. This is a limitation imposed by the matrix structure. Clearly, an extension is necessary if such Higgs scalars are to exist. We will again consider a Riemannian spin manifold extended by four points, this time with the algebra

$$\mathcal{A}_2 = M_2(C) \oplus M_2(C) \oplus M_2(C) \oplus M_2(C) \ . \tag{21}$$

The $U(4)$ sector is now introduced to the four point space by adding the auxiliary algebra $\mathcal{B}_2$, with right action on $\mathcal{H}$, given by

$$\mathcal{B}_2 = M_4(C) \oplus M_4(C) \oplus M_4(C) \oplus M_4(C) \ . \tag{22}$$

We make the same natural choice of vector bundle $\mathcal{F} = \mathcal{B}$, where $\mathcal{B} = \mathcal{A}_1 \otimes \mathcal{B}_2$. The physical Hilbert space can now be written as

$$\mathcal{P} = \mathcal{E} \otimes \mathcal{H} \otimes \mathcal{F} \ , \tag{23}$$

that is we have introduced a bi-module. Writing $\mathcal{H}_2^{\mathcal{A}}$ and $\mathcal{H}_2^{\mathcal{B}}$ for the Hilbert spaces corresponding to the algebras $\mathcal{A}_2$ and $\mathcal{B}_2$ respectively, $\mathcal{H}$ can be suggestively written as

$$\mathcal{H} = \mathcal{H}_2^{\mathcal{A}} \otimes L^2(S) \otimes \mathcal{H}_2^{\mathcal{B}} \ . \tag{24}$$

We will consider this as defining $U(2) \otimes U(4)$ gauge structure on each of the four copies of space–time as there is no reason, a priori, to assume that a single gauge symmetry only can be associated with each copy.

Corresponding to this extension we write down the generalized connection one-form

$$\rho = \sum_i a^i \, db^i \otimes 1 + 1 \otimes \sum_i A^i \, dB^i \; . \tag{25}$$

Note that the first 1 is a 4×4 unit matrix and the second a 2×2 unit matrix. We now introduce the Dirac operator

$$D = D_1 \otimes 1 \otimes 1 \otimes 1 + \gamma_5 \otimes D_2 \; , \tag{26}$$

where D_2 is given by

$$D_2 = \begin{pmatrix} 0 & m_{12} \otimes M_{12} \otimes K_{12} & m_{13} \otimes M_{13} \otimes K_{13} & m_{14} \otimes M_{14} \otimes K_{14} \\ m_{21} \otimes M_{21} \otimes K_{21} & 0 & m_{23} \otimes M_{23} \otimes K_{23} & m_{24} \otimes M_{24} \otimes K_{24} \\ m_{31} \otimes M_{31} \otimes K_{31} & m_{32} \otimes M_{32} \otimes K_{32} & 0 & m_{34} \otimes M_{34} \otimes K_{34} \\ m_{41} \otimes M_{41} \otimes K_{41} & m_{42} \otimes M_{42} \otimes K_{42} & m_{43} \otimes M_{43} \otimes K_{43} & 0 \end{pmatrix} \tag{27}$$

with m_{mn} the tree level vacuum expectation values in the $U(2)$ sector, M_{mn} the vacuum expectation values in the $U(4)$ sector and K_{mn} are 3×3 family mixing matrices. We will consider the construction of $\pi(\rho)$ for the general case first, introducing the relevant permutation symmetries between space–times once the form of the action has been established.

Since $d1 = 0$ we can re-express the connection ρ as

$$\rho = \sum_i (a^i \otimes 1) \, d(b^i \otimes 1) + \sum_i (1 \otimes A^i) \, d(1 \otimes B^i) \; . \tag{28}$$

The image of ρ under π is then

$$\begin{aligned} \pi(\rho) &= \sum_i (a^i \otimes 1) [D, b^i \otimes 1] + \sum_i (1 \otimes A^i) [D, 1 \otimes B^i] \\ &= \pi(\rho)_1 + \pi(\rho)_2 \; , \end{aligned} \tag{29}$$

where (suppressing the γ_5's for brevity)

$$\pi(\rho)_1 = \begin{pmatrix} A_2 & \phi_{12} \otimes M_{12} \otimes K_{12} & \phi_{13} \otimes M_{13} \otimes K_{13} & \phi_{14} \otimes M_{14} \otimes K_{14} \\ \phi_{21} \otimes M_{21} \otimes K_{21} & A_2 & \phi_{23} \otimes M_{23} \otimes K_{23} & \phi_{24} \otimes M_{24} \otimes K_{24} \\ \phi_{31} \otimes M_{31} \otimes K_{31} & \phi_{32} \otimes M_{32} \otimes K_{32} & A_2 & \phi_{34} \otimes M_{34} \otimes K_{34} \\ \phi_{41} \otimes M_{41} \otimes K_{41} & \phi_{42} \otimes M_{42} \otimes K_{42} & \phi_{43} \otimes M_{43} \otimes K_{43} & A_2 \end{pmatrix} \tag{30}$$

and

$$\pi(\rho)_2 = \begin{pmatrix} A_4 & m_{12}\otimes\Phi_{12}\otimes K_{12} & m_{13}\otimes\Phi_{13}\otimes K_{13} & m_{14}\otimes\Phi_{14}\otimes K_{14} \\ m_{21}\otimes\Phi_{21}\otimes K_{21} & A_4 & m_{23}\otimes\Phi_{23}\otimes K_{23} & m_{24}\otimes\Phi_{24}\otimes K_{24} \\ m_{31}\otimes\Phi_{31}\otimes K_{31} & m_{32}\otimes\Phi_{32}\otimes K_{32} & A_4 & m_{34}\otimes\Phi_{34}\otimes K_{34} \\ m_{41}\otimes\Phi_{41}\otimes K_{41} & m_{42}\otimes\Phi_{42}\otimes K_{42} & m_{43}\otimes\Phi_{43}\otimes K_{43} & A_4 \end{pmatrix} \tag{31}$$

with ϕ_{mn} and Φ_{mn} given by

$$\begin{aligned} \phi_{mn} &= \sum_i a^i_m (m_{mn} b^i_n - b^i_m m_{mn}) \\ \Phi_{mn} &= \sum_i A^i_m (M_{mn} B^i_n - B^i_m M_{mn}) \end{aligned} \tag{32}$$

The two form $d\rho$ will now be given by

$$d\rho = \sum_i d(a^i \otimes 1)\, d(b^i \otimes 1) + \sum_i d(1 \otimes A^i)\, d(1 \otimes B^i) \ , \tag{33}$$

with the image under π

$$\pi(d\rho) = \sum_i [D, a^i \otimes 1][D, b^i \otimes 1] + \sum_i [D, 1 \otimes A^i][D, 1 \otimes B^i] \ . \tag{34}$$

Gauge invariance of the spinor action $\langle \Psi, (D + \pi(\rho))\Psi \rangle$ under the transformation $\Psi \to {}^g\Psi = g\Psi$, where $g \in U(\mathcal{A}) \otimes U(\mathcal{B})$, demands that ρ transforms inhomogenously such that

$${}^g\rho = g\rho g^\dagger + g d g^\dagger \ , \tag{35}$$

where $g = g_2 \otimes g_4$. This can be written as

$$\begin{aligned} {}^g\rho = &\left\{ \sum_i (g_2 a^i) d(b^i g_2^\dagger) - g_2 (\sum_i a^i b^i - 1) d g_2^\dagger \right\} \otimes 1 \\ &+ 1 \otimes \left\{ \sum_i (g_4 A^i) d(B^i g_4^\dagger) - g_4 (\sum_i A^i B^i - 1) d g_4^\dagger \right\} \end{aligned} \tag{36}$$

so that gauge transformations can be defined directly on the constituent elements by

$$\begin{aligned} a^i \to {}^g a^i &= g_2 a^i \quad & A^i \to {}^g A^i &= g_4 A^i \\ b^i \to {}^g b^i &= b^i g_2^\dagger \quad & B^i \to {}^g B^i &= B^i g_4^\dagger \end{aligned} \tag{37}$$

if the constraints

$$\sum_i a^i b^i = 1 \quad \text{and} \quad \sum_i A^i B^i = 1 \tag{38}$$

are imposed.

Gauge transformations can be expressed in the representation π which from (35) take the form

$$\pi({}^g\rho) = g\pi(\rho)g^\dagger + g[D, g^\dagger] \tag{39}$$

and in component form become

$$\begin{aligned}
&{}^gA_2 = g_2A_2g_2^\dagger + g_2\not{\partial} g_2^\dagger\,, \quad {}^gA_4 = g_4A_4g_4^\dagger + g_4\not{\partial} g_4^\dagger\,, \quad m = 1,2,3,4\\
&{}^g(\phi_{mn}\otimes M_{mn} + m_{mn}\otimes \Phi_{mn} + m_{mn}\otimes M_{mn})\\
&\quad = g_2\otimes g_4(\phi_{mn}\otimes M_{mn} + m_{mn}\otimes \Phi_{mn} + m_{mn}\otimes M_{mn})g_2^\dagger\otimes g_4^\dagger\,, \quad m\neq n
\end{aligned} \tag{40}$$

Thus A_2 and A_4 are indeed the $U(2)$ and $U(4)$ gauge fields with the combination $(\phi_{mn}\otimes M_{mn} + m_{mn}\otimes\Phi_{mn} + m_{mn}\otimes M_{mn})$ scalar fields transforming covariantly, where ${}^g(m_{mn}\otimes M_{mn}) = m_{mn}\otimes M_{mn}$ in D. The form of the scalar fields demonstrates that ϕ_{mn} and Φ_{mn} represent independent fluctuations around the vacuum state specified by $m_{mn}\otimes M_{mn}$.

The representation of the curvature $\pi(\Theta)$ requires a determination of $\pi(d\rho)$ which, although a tedious calculation, is a direct generalization of the computation presented in [5]. Thus, rather than outlining the detailed procedure we will simply present the results. Expressed in terms of the gauge fields, Higgs fields and auxiliary fields the diagonal elements of the curvature can be written as

$$\begin{aligned}
\pi(\Theta)_{mm} &= \tfrac{1}{2}\gamma^{\mu\nu}F^{m(2)}_{\mu\nu} + \tfrac{1}{2}\gamma^{\mu\nu}F^{m(4)}_{\mu\nu}\\
&\quad + \sum_{p\neq m}(|K_{mp}|^2(|\phi_{mp}\otimes M_{mp} + m_{mp}\otimes\Phi_{mp}\\
&\quad + m_{mp}\otimes M_{mp}|^2 + |m_{mp}\otimes M_{mp}|^2)) - Y_m - X_{mm}
\end{aligned} \tag{41}$$

where

$$\begin{aligned}
X_{mm} &= \sum_i a^i_m\not{\partial}^2 b^i_m + \sum_i A^i_m\not{\partial}^2 B^i_m\\
&\quad + (\partial^\mu A^m_{2\mu} + A_2^{m\mu}A^m_{2\mu}) + (\partial^\mu A^m_{4\mu} + A_4^{m\mu}A^m_{4\mu}) - 2A_2^{m\mu}A^m_{4\mu}\,,\\
Y_m &= \sum_{p\neq m}\sum_i a^i_m|K_{mp}|^2|m_{mp}|^2\otimes|M_{mp}|^2 b^i_m + A^i_m|K_{mp}|^2|m_{mp}|^2\otimes|M_{mp}|^2 B^i_m\,,\\
F^{m(2)}_{\mu\nu} &= \partial_\mu A^m_{2\nu} - \partial_\nu A^m_{2\mu} + [A^m_{2\mu}, A^m_{2\nu}]\,,\\
F^{m(4)}_{\mu\nu} &= \partial_\mu A^m_{4\nu} - \partial_\nu A^m_{4\mu} + [A^m_{4\mu}, A^m_{4\nu}]
\end{aligned} \tag{42}$$

and, for example, $|K_{mp}|^2 = K_{mp}K_{pm}$. The non-diagonal elements of the curvature are given by $(m\neq n)$

$$
\begin{aligned}
\pi(\Theta)_{mn} = & -\gamma_5 K_{mn}(\not\partial(\phi_{mn}\otimes M_{mn} + m_{mn}\otimes\Phi_{mn}) \\
& + (A_2^m + A_4^m)(\phi_{mn}\otimes M_{mn} + m_{mn}\otimes\Phi_{mn} + m_{mn}\otimes M_{mn}) \\
& - (\phi_{mn}\otimes M_{mn} + m_{mn}\otimes\Phi_{mn} + m_{mn}\otimes M_{mn})(A_2^n + A_4^n)) \\
& + \sum_{p\neq m,n} K_{mp}K_{pn}((\phi_{mp}\otimes M_{mp} + m_{mp}\otimes\Phi_{mp} + m_{mp}\otimes M_{mp}) \\
& \times(\phi_{pn}\otimes M_{pn} + m_{pn}\otimes\Phi_{pn} + m_{pn}\otimes M_{pn}) \\
& - m_{mp}m_{pn}\otimes M_{mp}M_{pn}) - X_{mn}
\end{aligned}
\tag{43}
$$

where

$$
\begin{aligned}
X_{mn} = & \sum_i \sum_{p\neq m,n} K_{mp}K_{pn} \\
& \times\{a_m^i(m_{mp}m_{pn}\otimes M_{mp}M_{pn}b_n^i - b_m^i m_{mp}m_{pn}\otimes M_{mp}M_{pn}) \\
& + A_m^i(m_{mp}m_{pn}\otimes M_{mp}M_{pn}B_n^i - B_m^i m_{mp}m_{pn}\otimes M_{mp}M_{pn})\}.
\end{aligned}
\tag{44}
$$

Recall that we are working in four dimensional Euclidean space so that the gamma matrices employed satisfy: $\gamma_\mu^\dagger = -\gamma_\mu, \{\gamma_\mu, \gamma_\nu\} = -2\delta_{\mu\nu}$ and $\gamma_5 = \gamma_1\gamma_2\gamma_3\gamma_4$. Note also that the curvature is self-adjoint so that $\pi(\Theta)_{mn}^\dagger = \pi(\Theta)_{nm}$. The Euclidean space action can now be determined by exploiting (16) and takes the form:

$$
\begin{aligned}
I = & -\int d^4x \sum_m \mathrm{Tr}(\tfrac{1}{4}F_{\mu\nu}^{m(2)}F^{m(2)\mu\nu} + \tfrac{1}{4}F_{\mu\nu}^{m(4)}F^{m(4)\mu\nu} \\
& - \frac{1}{2}|\sum_{p\neq m}(|K_{mp}|^2(|\phi_{mp}\otimes M_{mp} + m_{mp}\otimes\Phi_{mp} + m_{mp}\otimes M_{mp}|^2 \\
& + |m_{mp}\otimes M_{mp}|^2)) - Y_m - X_{mm}|^2 \\
& + \frac{1}{2}\sum_{p\neq m,n}|K_{mp}|^2|(\partial_\mu(\phi_{mn}\otimes M_{mn} + m_{mn}\otimes\Phi_{mn}) \\
& + (A_{2\mu}^m + A_{4\mu}^m)(\phi_{mn}\otimes M_{mn} + m_{mn}\otimes\Phi_{mn} + m_{mn}\otimes M_{mn}) \\
& - (\phi_{mn}\otimes M_{mn} + m_{mn}\otimes\Phi_{mn} + m_{mn}\otimes M_{mn})(A_{2\mu}^n + A_{4\mu}^n))|^2 \\
& - \frac{1}{2}\sum_{n\neq m}\sum_{p\neq m,n}||K_{mp}|^2((\phi_{mp}\otimes M_{mp} + m_{mp}\otimes\Phi_{mp} + m_{mp}\otimes M_{mp}) \\
& \times(\phi_{pn}\otimes M_{pn} + m_{pn}\otimes\Phi_{pn} + m_{pn}\otimes M_{pn}) \\
& - m_{mp}m_{pn}\otimes M_{mp}M_{pn}) - X_{mn}|^2)
\end{aligned}
\tag{45}
$$

where we normalize the trace such that $\mathrm{Tr}\,1 = 1$. Note that since special unitary groups only will be considered, cross terms of the field strengths have been ignored.

We can now address ourselves to the construction of an $SU(2)_L\otimes SU(2)_R\otimes SU(4)$ partial unification model. In order to achieve two independent $U(2)$

gauge symmetries, rather than four, and to induce triplet Higgs fields, we will introduce the permutation symmetries

$$a_1^i = a_2^i, \quad a_3^i = a_4^i$$
$$b_1^i = b_2^i, \quad b_3^i = b_4^i \tag{46}$$

These are the identifications made by Chamseddine et al. [5] in their considerations on the left–right symmetric electroweak model. However, rather than a graded tracelessness condition on $\pi(\rho)_1$ we will simply impose the constraint

$$\mathrm{Tr}(\pi(\rho)_1) = 0 \ , \tag{47}$$

reducing $U(2)_L \otimes U(2)_R$ to $SU(2)_L \otimes SU(2)_R$, so avoiding the introduction of $U(1)$ factors. Since we want only one $U(4)$ field all the copies in $\pi(\rho)_2$ must be identified. If we were to choose identifications of the form (46) we would be considering a model with $U(4)_L \otimes U(4)_R$ gauge symmetry. The additional identifications which we must impose to avoid this are analogous to the criteria required to yield an $SO(10)$ rather than an $SU(16)$ symmetry in the $SO(10)$ models of Chamseddine and Fröhlich [7]. That is, we are comparing the case of a model with symmetry group $SU(2)_L \otimes SU(2)_R \otimes SU(4)_L \otimes SU(4)_R$, which is a subgroup of $SU(16)$, with the Pati–Salam partial unification which can be embedded as a maximal subgroup of $SO(10)$.

To achieve a Higgs structure which will allow for the generation of a large right handed neutrino mass at tree level we must consider, along with direct identifications between space–times, the inclusion of a conjugation symmetry. In this way we also introduce conjugate spinors into the spinor representation Ψ. This is to be contrasted with the electroweak case in which an additional conjugate set of fermions was needed to produce the full range of allowed Yukawa couplings. As $SO(1,3)$ and $SU(2)$ have conjugation matrices the conjugate spinors can be written as

$$\psi^c_{L,R} = i\tau_2 C \overline{\psi}^T_{L,R} \ , \tag{48}$$

where C is the Dirac conjugation matrix and $i\tau_2$ the $SU(2)$ conjugation matrix. Since we require complex representations in the $U(4)$ sector to be transformed to their complex conjugate, the charge conjugation operator must be an outer automorphism on the $U(4)$ algebra. The conjugation symmetries that we impose in the $U(4)$ sector, then, are given by

$$A_1^i \leftrightarrow A_2^{i*}, \quad A_3^{i*} \leftrightarrow A_4^i$$
$$B_1^i \leftrightarrow B_2^{i*}, \quad B_3^{i*} \leftrightarrow B_4^i \tag{49}$$

where A^*_m is the complex conjugate of A_m corresponding to the anti-representation, together with the identifications

$$A_1^i = A_4^i\,, \quad A_2^i = A_3^i$$
$$B_1^i = B_4^i\,, \quad B_2^i = B_3^i \tag{50}$$

so yielding a single $U(4)$ interaction.

Note that since all $SU(2)$ representations are real, the identifications made in (46) can be equivalently considered as conjugation symmetries. Thus, while all the space–time copies are identified in the $U(4)$ sector the $SU(2)$ sector remains to differentiate between the left and right regions of the model. Furthermore within each region, left and right, a consistent conjugation symmetry prevails between space-times. The added complication in the $U(4)$ sector is, as before, related to its chiral symmetry. The spinor Ψ can thus be written as

$$\Psi = \begin{pmatrix} \psi_L \\ i\tau_2 C\overline{\psi}_L^T \\ i\tau_2 C\overline{\psi}_R^T \\ \psi_R \end{pmatrix} \tag{51}$$

where the left and right handed assignments follow from imposing the chirality condition

$$(\gamma_5 \otimes \Gamma)\Psi = \Psi\ , \tag{52}$$

with $\Gamma = \mathrm{diag}(1,-1,1,-1)$, introduced after the Wick rotation to Minkowski space. Since all the elements in the $U(4)$ sector are identified we will reduce $U(4) \to SU(4)$ by simply imposing $\mathrm{Tr}(A_4) = 0$.

With our choice of symmetries between space–times the vector potential $\pi(\rho)_1$ takes the form (suppressing the family matrices K_{mn} and the γ_5's)

$$\pi(\rho)_1 = \begin{pmatrix} A_L & \Delta_L^{(1)} \otimes M_{12} & \phi' \otimes M_{13} & \phi \otimes M_{14} \\ \Delta_L^{(1)\dagger} \otimes M_{21} & \overline{A}_L & \phi^* \otimes M_{23} & \phi'^* \otimes M_{24} \\ \phi'^\dagger \otimes M_{31} & \phi^{*\dagger} \otimes M_{32} & \overline{A}_R & \Delta_R^{(1)\dagger} \otimes M_{34} \\ \phi^\dagger \otimes M_{41} & \phi'^{*\dagger} \otimes M_{42} & \Delta_R^{(1)} \otimes M_{43} & A_R \end{pmatrix} \tag{53}$$

where A_L and A_R are the $SU(2)_L$ and $SU(2)_R$ gauge fields, $\Delta_L^{(1)}$ and $\Delta_R^{(1)}$ are singlets and triplets in the respective groups and ϕ is a bi-doublet. Similarly, $\pi(\rho)_2$ takes the form

$$\pi(\rho)_2 = \begin{pmatrix} A_4 & m_{12} \otimes \Delta^{(2)} & m_{13} \otimes \Delta^{(2)} & m_{14} \otimes \Sigma \\ m_{21} \otimes \Delta^{(2)\dagger} & \overline{A}_4 & m_{23} \otimes \Sigma & m_{24} \otimes \Delta^{(2)\dagger} \\ m_{31} \otimes \Delta^{(2)\dagger} & m_{32} \otimes \Sigma & \overline{A}_4 & m_{34} \otimes \Delta^{(2)\dagger} \\ m_{14} \otimes \Sigma & m_{42} \otimes \Delta^{(2)} & m_{43} \otimes \Delta^{(2)} & A_4 \end{pmatrix} \tag{54}$$

From the space–time symmetries the constituent fields transform as

$$\Delta^{(2)} \sim \mathbf{4} \times \mathbf{4} = \mathbf{6} + \mathbf{10}$$
$$\Sigma \sim \mathbf{4} \times \overline{\mathbf{4}} = \mathbf{1} + \mathbf{15} \tag{55}$$

under $SU(4)$. We know the covariant form taken by Higgs scalars under a general gauge transformation from (40). Consequently, the Higgs fields which enter the model will transform under $SU(2)_L \otimes SU(2)_R \otimes SU(4)$ as:

$$\begin{aligned}
\Delta_L &= \Delta_L^{(1)} \otimes M_{12} + m_{12} \otimes \Delta^{(2)} \\
&\sim (\mathbf{3},\mathbf{1},\mathbf{6}) + (\mathbf{3},\mathbf{1},\mathbf{10}) + (\mathbf{1},\mathbf{1},\mathbf{6}) + (\mathbf{1},\mathbf{1},\mathbf{10}) \\
\Delta_R &= \Delta_R^{(1)} \otimes M_{43} + m_{43} \otimes \Delta^{(2)} \\
&\sim (\mathbf{1},\mathbf{3},\mathbf{6}) + (\mathbf{1},\mathbf{3},\mathbf{10}) + (\mathbf{1},\mathbf{1},\mathbf{6}) + (\mathbf{1},\mathbf{1},\mathbf{10}) \\
\Phi &= \phi \otimes M_{14} + m_{14} \otimes \Sigma \\
&\sim (\mathbf{2},\mathbf{2},\mathbf{1}) + (\mathbf{2},\mathbf{2},\mathbf{15}) \\
\Phi' &= \phi' \otimes M_{13} + m_{13} \otimes \Delta^{(2)} \\
&\sim (\mathbf{2},\mathbf{2},\mathbf{6}) + (\mathbf{2},\mathbf{2},\mathbf{10}).
\end{aligned} \tag{56}$$

The other entries follow from the inter-space–time symmetries. This in turn breaks the Higgs field degeneracy in the vector potential. Embedding our partial unification model into $SO(10)$ we see immediately that our Higgs fields transform as components of relevant Higgs representations often chosen for such models, e.g. **10**, **120**, **126** and **210**. We could easily obtain other components by different choices of symmetries between space–times. Nevertheless, with the choice of symmetries taken we have generated the Higgs components which are required to have non-zero vacuum expectation values for a non-minimal model.

With the product structure between the $SU(2)$ and $SU(4)$ sectors made manifest in the form of the Higgs scalars we must correspondingly realize this mathematical structure in the spinors, which take the form

$$\psi_{L,R} = \begin{bmatrix} u \\ d \end{bmatrix} \otimes (a,b,c,d) \tag{57}$$

where the column $[u,d]$ indicates valency and the spin-zero row (a,b,c,d) indicates colour degrees of freedom [8]. This is to be compared with the physical realization given by (20). From the spinor action $\langle \Psi, (D + \pi(\rho))\Psi \rangle$ we see that we can generate the Yukawa couplings:

$$\begin{aligned}
\mathcal{L}_Y &= K_{14}\overline{\Psi}_L(\Phi + \langle \Phi \rangle)\psi_R + K_{32}\overline{\Psi}_L\tau_2(\Phi^* + \langle \Phi^* \rangle)^\dagger \tau_2 \psi_R \\
&- iK_{21}\psi_L^T \tau_2 C^{-1}(\Delta_L + \langle \Delta_L \rangle)^\dagger \psi_L - iK_{34}\psi_R^T \tau_2 C^{-1}(\Delta_R + \langle \Delta_R \rangle)^\dagger \psi_R \\
&+ \text{H.c.}
\end{aligned} \tag{58}$$

where the Hermitian conjugates emerge automatically from the self-adjointness of $\pi(\rho)$. The coupling to Φ will again be responsible for the usual quark and lepton masses. However, we now have a tree level coupling which can yield large right handed neutrino masses by the see-saw mechanism, i.e. the coupling producing Majorana masses given by the components $(\mathbf{3},\mathbf{1},\mathbf{10})$ and $(\mathbf{1},\mathbf{3},\mathbf{10})$ of Δ_L and Δ_R respectively [8]. Coupling to the conjugate bi-doublet Higgs is a natural consequence of the introduction of conjugation symmetries.

The true viability of the model is dependent on the survival of the Higgs potential once suitable vacuum expectation values for the Higgs scalars have been chosen. This corresponds to the elimination of unwanted components from (56), where we will take the only components with non-zero vacuum expectation values to be $\Delta_L \sim (\mathbf{3},\mathbf{1},\mathbf{10})$, $\Delta_R \sim (\mathbf{1},\mathbf{3},\mathbf{10})$ and $\Phi \sim (\mathbf{2},\mathbf{2},\mathbf{1})$. The symmetry breaking scheme will then take the form:

$$\begin{aligned} SU(2)_L \otimes SU(2)_R \otimes SU(4) &\xrightarrow{\langle \Delta_R \rangle = v_R} SU(2)_L \otimes SU(3)_C \otimes U(1)_Y \\ &\xrightarrow{\langle \Phi \rangle \neq 0} SU(3)_C \otimes U(1)_Q \end{aligned}$$

where we have the expectation value hierarchy $\langle \Delta_L \rangle \ll \langle \Phi \rangle \ll \langle \Delta_R \rangle$. The form of the vacuum expectation values is dictated by the requirement that $U(1)_Q$ survive so that only charge zero components can be non-zero. For a fractionally charged quark model in which the gluons also remain chargeless the charge operator Q is given by [8]

$$Q = I_{3L} + I_{3R} + 1/2 \begin{pmatrix} 1/3 & 0 & 0 & 0 \\ 0 & 1/3 & 0 & 0 \\ 0 & 0 & 1/3 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{59}$$

so that the Higgs vacuum expectation values become

$$\langle \Delta_R \rangle = v_R \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = v_R S_2 \otimes S_4$$

$$\langle \Delta_L \rangle = v_L \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = v_L S_2 \otimes S_4$$

$$\langle \Phi \rangle = \begin{pmatrix} u_1 & 0 \\ 0 & u_2 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} u_1 & 0 \\ 0 & u_2 \end{pmatrix} \otimes I_4 \tag{60}$$

We can now determine the independent contributions from the auxiliary fields which must be eliminated. The X and Y fields are given by

$$
\begin{aligned}
Y_1 = {} & |K_{12}|^2 (\sum_i a_1^i |v_L|^2 S_2 S_2^\dagger \otimes S_4 b_1^i + A^i |v_L|^2 S_2 S_2^\dagger \otimes S_4 B^i) \\
& + |K_{14}|^2 (\sum_i a_1^i \begin{pmatrix} |u_1|^2 & 0 \\ 0 & |u_2|^2 \end{pmatrix} \otimes I_4 b_1^i \\
& + A^i \begin{pmatrix} |u_1|^2 & 0 \\ 0 & |u_2|^2 \end{pmatrix} \otimes I_4 B^i) \\
Y_3 = {} & |K_{34}|^2 (\sum_i a_3^i |v_R|^2 S_2^\dagger S_2 \otimes S_4 b_3^i + A^{i*} |v_R|^2 S_2^\dagger S_2 \otimes S_4 B^{i*}) \\
& + |K_{32}|^2 (\sum_i a_3^i \begin{pmatrix} |u_1|^2 & 0 \\ 0 & |u_2|^2 \end{pmatrix} \otimes I_4 b_3^i \\
& + A^{i*} \begin{pmatrix} |u_1|^2 & 0 \\ 0 & |u_2|^2 \end{pmatrix} \otimes I_4 B^{i*}) \\
X_{12} = {} & X_{34} = X_{23} = X_{14} = 0 \\
X_{13} = {} & \sum_i K_{12} K_{23} \{ a_1^i v_L u_2 (S_2 \otimes S_4 b_3^i - b_1^i S_2 \otimes S_4) \\
& + A^i v_L u_2 (S_2 \otimes S_4 B^{i*} - B^i S_2 \otimes S_4) \} \\
& + K_{14} K_{43} \{ a_1^i v_R u_1 (S_2 \otimes S_4 b_3^i - b_1^i S_2 \otimes S_4) \\
& + A^i v_R u_1 (S_2 \otimes S_4 B^{i*} - B^i S_2 \otimes S_4) \}
\end{aligned} \tag{61}
$$

where the others follow from the permutation symmetries. Clearly, the field X_{13} is auxiliary and thus must be eliminated . The independent contributions from the Y fields can be easily found where, for example, Y_1 can be rewritten as

$$
\begin{aligned}
Y_1 = {} & |K_{12}|^2 (2|v_L|^2 S_2 S_2^\dagger \otimes S_4 - \left(\sum_i a_1^i |v_L|^2 (S_2^\dagger S_2 \otimes S_4 b_1^i - b_1^i S_2^\dagger S_2 \otimes S_4) \right) \\
& - (\sum_i A^i |v_L|^2 (S_2 S_2^\dagger \otimes \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} B^i - B^i S_2 S_2^\dagger \otimes \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}))) \\
& + |K_{14}|^2 (2 \begin{pmatrix} |u_1|^2 & 0 \\ 0 & |u_2|^2 \end{pmatrix} \otimes I_4 \\
& - (\sum_i a_1^i (\begin{pmatrix} |u_2|^2 & 0 \\ 0 & |u_1|^2 \end{pmatrix} \otimes I_4 b_1^i - b_1^i \begin{pmatrix} |u_2|^2 & 0 \\ 0 & |u_1|^2 \end{pmatrix} \otimes I_4)))
\end{aligned}
$$

with the others following similarly. We therefore have the non-independent contributions

$$2\left(|K_{12}|^2|v_L|^2 S_2 S_2^\dagger \otimes S_4 + |K_{14}|^2 \begin{pmatrix} |u_1|^2 & 0 \\ 0 & |u_2|^2 \end{pmatrix} \otimes I_4\right)$$
$$= 2\left(|K_{12}|^2|m_{12} \otimes M_{12}|^2 + |K_{14}|^2|m_{14} \otimes M_{14}|^2\right) \tag{62}$$

where the remainder is eliminated. Note that the X_{mm} are also eliminated. The Higgs potential can thus be shown to take the form (where an orthogonality condition on the components has been imposed to simplify the calculations)

$$\begin{aligned} V = {} & (\mathrm{Tr}\,|K_{12}|^4 - (\mathrm{Tr}\,|K_{12}|^2)^2)||\Delta_L + m_{12} \otimes M_{12}|^2 - |m_{12} \otimes M_{12}|^2|^2 \\ & + (\mathrm{Tr}\,|K_{14}|^4 - (\mathrm{Tr}\,|K_{14}|^2)^2)||\Phi + m_{14} \otimes M_{14}|^2 - |m_{14} \otimes M_{14}|^2|^2 \\ & + (\mathrm{Tr}\,|K_{23}|^4 - (\mathrm{Tr}\,|K_{23}|^2)^2)||\Phi^* + m_{23} \otimes M_{23}|^2 - |m_{23} \otimes M_{23}|^2|^2 \\ & + (\mathrm{Tr}\,|K_{43}|^4 - (\mathrm{Tr}\,|K_{43}|^2)^2)||\Delta_R + m_{43} \otimes M_{43}|^2 - |m_{43} \otimes M_{43}|^2|^2 \end{aligned} \tag{63}$$

and thus survives. To yield this potential the independent contributions from the Y's have been modded out by hand rather than via the equations of motion. This is necessary since the potential will still otherwise vanish, even though the Y's yield non-independent contributions. The cause of this is our over simplification in choice of vacuum expectation values by setting the Φ' contribution to zero. Including a small non-zero expectation value for the $(\mathbf{2},\mathbf{2},\mathbf{10})$ component will not affect the symmetry breaking pattern nor the fermion sector since it decouples by the requirement of Lorentz invariance. However, this will now give non-trivial contributions from the X_{mn}'s so that the full potential can be arranged to survive.

Looking more closely at the fermionic action it follows from the quark–lepton unification that the same family mixing matrix will operate on the u and d quarks. This does not occur in the $SU(5)$ and standard model examples previously considered since an additional set of spinors must be introduced so that the u quark may attain a non-zero mass [5]. This also allows for different mixing matrices and thus the existence of a Cabibbo angle. Quark–lepton unification eliminates the need to introduce an additional set of spinors and also, therefore, different mixing matrices. This was the dilemma faced in the $SO(10)$ model of Chamseddine and Fröhlich [7] for which introducing singlet spinors resolved the problem. Compelled by the success of the model we have constructed so far we will take a different approach to this problem. For models without a right handed neutrino no mixing occurs in the neutrino sector because the mass matrix is identically zero. Quark–lepton unification implies breaking this neutrino degeneracy. Note also that we now have an additional degree of freedom from introducing a bi-module so that family mixing for quarks and leptons may be differentiated. We will thus extend the mixing matrices to take the form

$$K_{mn} = \mathrm{diag}(f_{\alpha\beta}, f'_{\alpha\beta})_{mn} \ , \tag{64}$$

acting on d^α and u^α in the multiplet structure of (20), where α refers to the three families. This is analogous to the additional mixing allowed in other models by giving the u quarks non-zero masses as well as providing mixing among neutrinos. In this way u quarks and neutrinos attain additional structure on the same footing, which is consistent with the concept of neutrinos being the fourth up quark.

It is straightforward now to write down the full fermionic and bosonic action from (13) and (45), where consistent normalizations for the kinetic energies can be accomodated by an appropriate rescaling. Since we have a bi-module structure separating the introduction of $SU(2)$ and $SU(4)$ we are free to implement different coupling strengths in these sectors. We thus have an acceptable model for which many phenomenonlogical parameters may be adjusted to yield results close to the experimental values. Importantly, we have avoided extending our Higgs sector beyond that needed to produce the required symmetry breaking pattern.

5. Conclusion

By generalizing previous approaches to include a non-trivial extension by way of a bi-module we have been able to formulate a model which can yield tree level masses to neutrinos, avoiding the quantization problem and the inclusion of exotic fermions. Furthermore, we have broken the Higgs field degeneracy inherent in the vector potential thus enlarging our choice of scalar fields. While not the simplest approach to the introduction of neutrino masses, left–right symmetric models of this kind have the advantage of providing the freedom to incorporate important phenomenological features by the inclusion of intermediate scales. Such symmetry breaking scales, corresponding to the "distance" between copies of space–time, can now find a geometrical basis. Thus, in the absence of a quantization mechanism or embedding of supersymmetry into non-commutative geometry our approach yields a consistent formulation as well as utilizing more fully the freedom provided by this mathematical framework.

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References

[1] P. Forgacs and N.S. Manton, Space–time symmetries in gauge theories, Comm. Math. Phys. 72 (1980) 15;
G. Chapline and R. Slansky, Dimensional reduction and flavor chirality, Nucl. Phys. B 209 (1982) 461;
N.S. Manton, Fermions and parity violation in dimensional reduction schemes, Nucl. Phys. B 193 (1981) 502;
G. Chapline and N.S. Manton, The geometrical significance of certain Higgs potentials: an approach to grand unification, Nucl. Phys. B 184 (1981) 391.

[2] A. Connes and J. Lott, Particle models and noncommutative geometry, Nucl. Phys. B (Proc. Suppl.) 18 (1991) 29.

[3] A. Connes and J. Lott, The metric aspect of non-commutative geometry, in: *Proceedings of 1991 Cargese Summer Conference*, eds. J. Fröhlich et al. (Plenum, 1992).

[4] A. Connes, Essay on physics and non-commutative geometry, in: *The Interface of Mathematics and Particle Physics*, Eds. D. Quillen, G. Segal and S. Tsou (Oxford University Press, 1990); Non-commutative differential geometry, Publ. Math IHES 62 (1981) 41.

[5] A.H. Chamseddine, G. Felder and J. Fröhlich, Unified gauge theories in non-commutative geometry, Phys. Lett. B 296 (1993) 109; Grand unification in non-commutative geometry, Nucl. Phys. B 395 (1993) 672.

[6] J.C. Pati, Grand unification with $SU(4)$ color, $\sin^2\theta_W$ and proton decay searches, in: *Eighth Workshop on Grand Unification*, Ed. K.C. Wali (World Scientific, Singapore, 1988).

[7] A.H. Chamseddine and J. Fröhlich, $SO(10)$ unification in non-commutative geometry, Commun. Math. Phys. 155 (1993) 205.

[8] R.N. Mohapatra and R.E. Marshak, Local $B - L$ symmetry of electroweak interactions, Majorana neutrinos and neutron oscillations, Phys. Rev. Lett. 44 (1980) 1316;
J.C. Pati and A. Salam, Lepton number as the fourth "color", Phys. Rev. D 10 (1974) 275.

[9] R. Coquereaux, G. Esposito-Farese and G. Vaillant, Higgs fields as Yang–Mills fields and discrete symmetries, Nucl. Phys. B 353 (1991) 689.

[10] M. Dubois-Violette, R. Kerner and J. Madore, Classical bosons in a non-commutative geometry, Class. Quant. Grav. 6 (1989) 1709; Noncommutative differential geometry and new models of gauge theory, J. Math. Phys. 31 (1990) 323.

[11] B.S. Balakrishna, F. Gürsey and K.C. Wali, Towards a unified treatment of Yang–Mills and Higgs fields, Phys. Rev. D 44 (1991) 3313; Noncommutative geometry and Higgs mechanism in the Standard Model, Phys. Lett. B 254 (1991) 430.

[12] B.E. Hanlon and G.C. Joshi, A noncommutative geometric approach to left–right symmetric weak interactions, Lett. Math. Phys. 27 (1993) 105.

[13] E. Witten, Neutrino masses in the minimal $O(10)$ theory, Phys. Lett. B 91 (1980) 81.

[14] J. Choi and R.R. Volkas, Exotic fermions in the left–right symmetric model, Phys. Rev. D 45 (1992) 4261.